J. Fluid Mech. (1973), *vol.* 59, *part* 2, *pp.* 263–279
Printed in Great Britain

On Burgers' model equations for turbulence

By J. D. MURRAY
Mathematical Institute, Oxford University

(Received 11 October 1972)

Burgers' (1939) model equations for turbulence are considered analytically using a singular perturbation and nonlinear wave approach. The results indicate that there is an ultimate steady turbulent state. This is in agreement with the numerical results of Lee (1971) but not with Case & Chiu (1969): the last two papers start with a Fourier series approach.

A consequence of this model is that small disturbances ultimately grow into a single large domain of relatively smooth flow, accompanied by a vortex sheet in which strong vorticity is concentrated. This makes the results from the model different from those usually expected for turbulent flow fields. The model, as a result of its simplicity, has retained a degree of regularity which is not found in most forms of turbulence.

1. Introduction and the model equations

In an attempt to investigate some of the effects on turbulence of the viscous and nonlinear terms in the Navier–Stokes equations and to illustrate mathematically such turbulent phenomena as dissipation layers, energy balance and so on, Burgers (1939, see also 1948) suggested a model set of equations which are simpler than the Navier–Stokes ones. The specific problems for which he suggested that the model applies are turbulent flows in straight channels under the action of a constant pressure drop. Here we consider his one-dimensional equations, which are

$$v_t + 2vv_y - \frac{U}{h}v = \nu v_{yy}, \tag{1}$$

$$hU_t = P - \frac{\nu}{h}U - \frac{1}{h}\int_0^h v^2\,dy, \tag{2}$$

where $v(y,t)$ is the velocity of the turbulent motion, $U(t)$ the velocity of the primary or mean motion, P the analogue of the external force (pressure) acting on the primary motion, ν the kinematic viscosity, h the channel width and y, where $0 \leqslant y \leqslant h$, the co-ordinate across the channel. In (1), Uv/h represents a nonlinear transmission of energy from the primary to the secondary motion. In (2), $\nu U/h$ is the viscous force on the primary motion. Boundary and initial conditions for (1) and (2) are zero turbulent velocity at the walls,

$$v(0,t) = 0 = v(h,t),$$

and $v(y,0)$ and $U(0)$ given. A motivation for the model equations (1) and (2) is seen on adding U times (2) to (1) multiplied by v and integrated with respect to y from $y = 0$ to $y = h$, which gives an energy balance

$$\frac{d}{dt}\left[\tfrac{1}{2}hU^2+\frac{1}{2}\int_0^h v^2\,dy\right] = PU-\frac{\nu U^2}{h}-\nu\int_0^h v_y^2\,dy. \tag{3}$$

This says that the rate of change of the total kinetic energy of the motion equals the work done by the external force on the primary motion, that is, PU less the viscous dissipation $\nu U^2/h$ in the primary motion, less the viscous dissipation $\nu\int_0^h v_y^2\,dy$ in the turbulent motion. The nonlinear term Uv/h in (1) does not appear in (3) since it represents an internal process which effects energy exchange.

Introduce non-dimensional quantities, denoted by primes, by

$$v' = v/P^{\frac{1}{2}},\quad U' = U/P^{\frac{1}{2}},\quad y' = y/h,\quad t' = tP^{\frac{1}{2}}/h,\quad R = P^{\frac{1}{2}}h/\nu = 1/\epsilon, \tag{4}$$

where R is a Reynolds number. Substitution of (4) into (1) and (2) gives the dimensionless form of the equations as

$$v_t+2vv_y-Uv = \epsilon v_{yy}, \tag{5}$$

$$U_t+\epsilon U = 1-\int_0^1 v^2\,dy, \tag{6}$$

where for convenience we have omitted the primes; it is to be understood in what follows that all quantities are dimensionless. Boundary and initial conditions for (5) and (6) are

$$v(0,t) = 0 = v(1,t),\quad v(y,0) = f(y),\quad U(0) = U_0, \tag{7}$$

where $f(0) = 0 = f(1)$ and $U_0 > 0$.

Burgers (1939, 1948) studied the steady-state solutions of (5) and (6) and as a preliminary considered the unsteady state by setting

$$v(y,t) = \sum_{n=1}^{\infty} \xi_n(t)\sin n\pi y, \tag{8}$$

which satisfies the boundary conditions on v, the first two of conditions (7). Substitution of (8) into (5) and (6) gives the coupled nonlinear ordinary differential equations for $\xi_n(t)$ and $U(t)$ as

$$\frac{d\xi_n}{dt} = (U-\epsilon n^2\pi^2)\xi_n-n\pi\left[\frac{1}{2}\sum_{k=1}^{n-1}\xi_k\xi_{n-k}-\sum_{k=1}^{\infty}\xi_k\xi_{n+k}\right], \tag{9}$$

$$\frac{dU}{dt} = 1-\epsilon U-\frac{1}{2}\sum_{k=1}^{\infty}\xi_n^2. \tag{10}$$

If we linearize (5) and (6) about the steady state $v = 0$, $U = 1/\epsilon$, which is called the laminar solution of (5) and (6), we get, in place of (9),

$$d\xi_n/dt = [(1/\epsilon)-\epsilon n^2\pi^2]\xi_n,\quad \text{so that}\quad \xi_n \propto \exp[\epsilon^{-1}(1-\epsilon^2\pi^2n^2)t], \tag{11}$$

where the ξ_n are the small disturbance functions which will give the perturbation in v from (8). From (11) we see that linear instability occurs when

$$1/R = \epsilon < 1/\pi = \epsilon_c = 1/R_c, \tag{12}$$

where ϵ_c, the critical value of ϵ $(= 1/R)$, is defined by (12). This paper will be concerned with equations (5) and (6) for values of $\epsilon < \epsilon_c$.

Case & Chiu (1969), following Burgers (1939), used the above Fourier series approach and studied in detail the stability of truncated forms of (9) and (10). Specifically, they considered two sets of these equations, namely those truncated at $n = 2$ and at $n = 3$. They found that, as expected, a transition from the laminar state to a turbulent state occurs when $R = R_c$. They further found, using their truncated equations, that it is possible to have transitions from one turbulent state to another depending on the specific range of R ($> R_c$). They also concluded from their analysis that it is possible to cause a finite jump from one turbulent state to another by varying the Reynolds number by an infinitesimal amount.

Lee (1971) also used expansion (8) for $v(y, t)$ and considered the quasi-steady form of (9) and (10), which consists of (9) as it stands but with dU/dt omitted in (10), which becomes

$$U = \frac{1}{\epsilon}\left[1 - \frac{1}{2}\sum_{k=1}^{\infty} \xi_n^2\right]. \tag{13}$$

Lee's purpose was to simulate the initial-value problem for (9) and (10). He also reinvestigated some of the results and conjectures of Case & Chiu (1969). For a necessarily restricted class of initial data he computed, numerically, the solutions using (9) and (13) truncated at an n sufficiently high that the truncation error $|\xi_n/\xi_1| < 3 \times 10^{-3}$: with $R = 4$ ($\epsilon = 0{\cdot}25$), $n = 21$ was required. Lee concluded from his analysis that (9) and (10) give degenerate turbulent states when truncated to the order used by Case & Chiu (1969) and that the finite turbulent–turbulent transitions they predicted are not realizable. He further concluded that the truncated form of (9) and (10), with n sufficiently large, yields a unique equilibrium state which depends on R and which is obtained from members of the specific class of initial data taken. This result is also at variance with Case & Chiu's (1969) analysis.

In this paper we consider (5) and (6) analytically by exploiting the fact that ϵ ($= 1/R$) is a small parameter when $\epsilon < \epsilon_c = 1/\pi$. We use a singular perturbation procedure which avoids the problems involved with the Fourier series approach. One result indicated by the analysis is that there is, for (5) and (6), a unique equilibrium steady state. This is in agreement with Lee (1971). Here, however, we need not restrict the class of initial data. We also conclude that there are no turbulent–turbulent transitions as found by Case & Chiu (1969) with the truncated forms of (9) and (10). Perhaps the most important consequence of the results derived here is that they suggest that Burgers' model equations imply that small initial turbulent disturbances ultimately grow into large domains with relatively smooth interior flow, accompanied by narrow boundary or transition layers, separating a domain from its neighbours or from the walls of the flow field. The growth of these large domains represents the generation of lower Fourier modes; that is, 'coarse-grained turbulence'. The formation of the thin transition layers represents the excitation of high Fourier modes. There is thus 'fine-grained turbulence' – that is, a strong vortex motion – but it is concentrated into thin layers. In this respect the model yields features different from those usually expected for turbulence flow fields (see in this connexion §4 below).

2. Steady state: singular perturbation solutions

The steady-state form of (5) and (6) with (7) gives $v = v(y)$ and U as solutions of

$$\epsilon v_{yy} - 2vv_y + Uv = 0, \quad v(0) = 0 = v(1), \tag{14}$$

$$U = \frac{1}{\epsilon} - \frac{1}{\epsilon}\int_0^1 v^2\, dy. \tag{15}$$

Burgers (1939) studied (14) and obtained solutions in the form

$$v = \pm (2U\epsilon)^{\frac{1}{2}} [C - \eta + \log(1+\eta)]^{\frac{1}{2}}, \tag{16a}$$

$$y = \pm \left(\frac{\epsilon}{2U}\right)^{\frac{1}{2}} \int_{\eta_1}^{\eta_2} (1+\eta)^{-1} [C - \eta + \log(1+\eta)]^{-\frac{1}{2}}\, d\eta, \tag{16b}$$

where $\eta = -2v_y/U$ and $\eta_1 < 0$ and $\eta_2 > 0$ are the roots of $C - \eta + \log(1+\eta) = 0$, where $C > 0$ is a constant to be determined. We follow Burgers' discussion. From (16a) it follows that $v = 0$ when $\eta = \eta_1$ and $\eta = \eta_2$. If $\eta = \eta_1$ is associated with $y = 0$, $\eta = \eta_2$ may be associated with $v = 0$ at some specified $y > 0$ which determines C. When η_2 is associated with $y = 1$ denote this C by C_1. Equation (16b) with $y = 1$ and $C = C_1$ then gives C_1 implicitly as a function of the corresponding U, U_1 say. To get U_1, we would then have to use (15) with v given by (16a) with $C = C_1(U_1)$. Such a procedure is analytically not possible. The solution v, $v_1(y)$, say, is, however, similar to that in figure 1 (a) below. If $\eta = \eta_2$ is now associated with $y = \frac{1}{2}$, say, we get another solution, $v_2(y)$, with $C = C_2(U_2)$ given by (16b) on setting $y = \frac{1}{2}$. This solution gives v for $0 \leqslant y \leqslant \frac{1}{2}$ and the solution for $\frac{1}{2} \leqslant y \leqslant 1$ is obtained from the $0 \leqslant y \leqslant \frac{1}{2}$ form on substituting $1 - y$ for y and $-v$ for v: equation (14) is invariant under this transformation. Figure 1 (b) illustrates $v_2(y)$. In this way other solutions are shown to exist for different C. The condition that there be $m - 1$ nodes in $0 < y < 1$ in the solution is, from (16b),

$$\frac{1}{m} = \left(\frac{\epsilon}{2U_m}\right)^{\frac{1}{2}} \int_{\eta_1}^{\eta_2} (1+\eta)^{-1} [C_m - \eta + \log(1+\eta)]^{-\frac{1}{2}}\, d\eta. \tag{17}$$

This is illustrated in figure 1 (c). Burgers (1939) showed that $\{C_m\}$ is a decreasing sequence as m increases, and as $C_m \rightarrow 0$ equation (17) becomes simply

$$1/m = \pi(\epsilon/U_m)^{\frac{1}{2}}.$$

Since U_m has a maximum of $1/\epsilon$ from (15) this last expression gives an upper limit to m, say M, where $M = [1/\pi\epsilon]$ is the largest integer less than or equal to $1/\pi\epsilon$. There is thus a finite number of possible solutions for $v(y)$ which have either 0, 1, ..., or M nodes.

In view of the implicit analytical difficulties in the solutions (16) with (17) we obtain here specific expressions for the dissipation

$$\int_0^1 v^2\, dy$$

and the corresponding $U = U_m$ for each nodal solution. We reconsider (14) and (15) and obtain singular perturbation solutions for ϵ small ($\epsilon < 1/\pi$ is sufficiently small). We also find the mean flow U_m for each solution.

It is helpful below to use an alternative form for

$$\int_0^1 v^2\,dy$$

obtained by multiplying (14) by v and integrating from 0 to 1 to get

$$\epsilon\int_0^1 v_{yy}v\,dy-[\tfrac{2}{3}v^3]_0^1+U\int_0^1 v^2\,dy=0$$

and so

$$\phi=\int_0^1 v^2\,dy=\frac{\epsilon}{U}\int_0^1 v_y^2\,dy, \tag{18}$$

where for convenience we have written ϕ as the turbulent dissipation. With (18), (15) becomes

$$U=\frac{1}{\epsilon}(1-\phi)=\frac{1}{\epsilon}-\frac{1}{U}\int_0^1 v_y^2\,dy. \tag{19}$$

To be more precise, ϕ is the Reynolds shear stress and (19) states that the mean flow is being modified by the turbulent fluctuations via the Reynolds stress term.

In the absence of shock or steep regions ϵv_{yy} is of $O(\epsilon)$ and non-singular perturbation solutions of (14) are simply straight lines obtained from

$$\left.\begin{aligned}-2vv_y+Uv&=\epsilon v_{yy}=O(\epsilon),\\ v&=\tfrac{1}{2}Uy+A+o(\epsilon^N)\quad\text{for all } N,\end{aligned}\right\} \tag{20}$$

so that

where A is an arbitrary constant. We cannot satisfy both boundary conditions in (14) and so we must have at least one singular region. If it is at $y=\alpha$, say, then in the singular region we write $\xi=(y-\alpha)/\epsilon$ and (14) becomes

$$\epsilon^{-1}v_{\xi\xi}-\epsilon^{-1}2vv_\xi-Uv=0,$$

which to $O(1)$ has solutions

$$v(\xi)=a\tanh a(b'-\xi),\quad v(y)=a\tanh[a\epsilon^{-1}(b-y)], \tag{21}$$

where a and b are constants ($b'=\epsilon^{-1}(b-\alpha)$). Uniformly valid asymptotic solutions of (14) to $O(1)$ for $\epsilon\ll 1$ are, from (20) and (21),

$$v(y)=\tfrac{1}{2}Uy+A+a\tanh[a\epsilon^{-1}(b-y)], \tag{22}$$

as given by Burgers (1939).

It might appear to be possible to have an infinity of solutions since we might take (20) as any straight line simply connected to another straight line by a singular solution (21). However, since $a\tanh[a\epsilon^{-1}(b-y)]\sim\pm a$ as $(b-y)/\epsilon\to\pm\infty$ we can only have symmetric and similar steep regions at all points in the interior $0<y<1$. Furthermore all solutions must have an *integral* number of nodes with the same form for the solution between each node (compare with figures 1 (*b*) and (*c*)). To see this we consider the phase plane for (14), which is obtained from

$$w=v_y,\quad dw/dv=v(2w-U)/\epsilon w \tag{23}$$

and is as in figure 2. Because of the symmetry of the integral curves for (23) the maximum $|v|$ is the same for the solution between each node, where $v=0$.

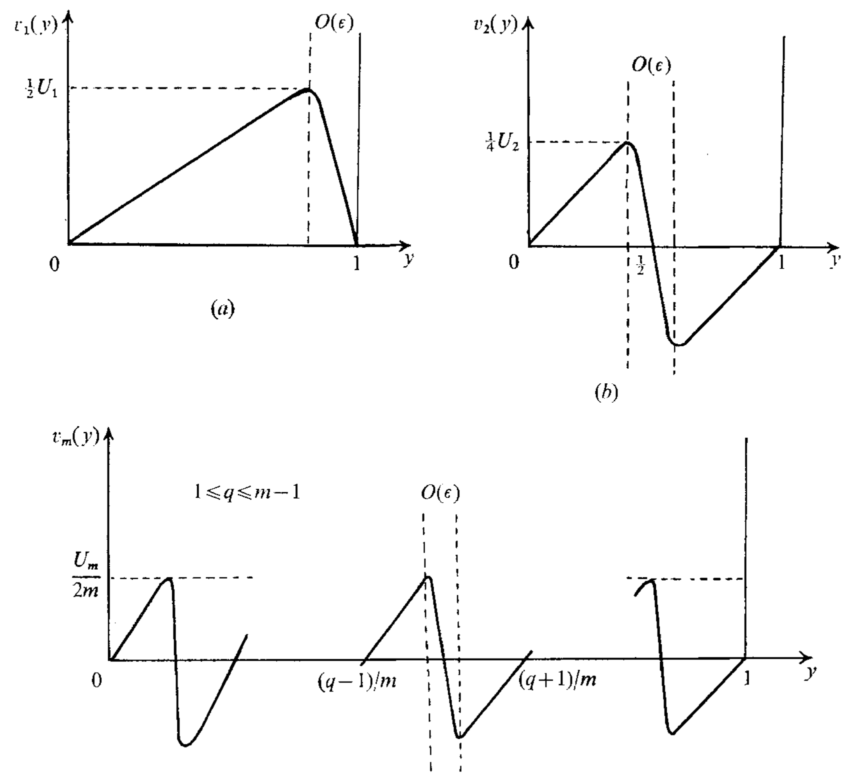

FIGURE 1

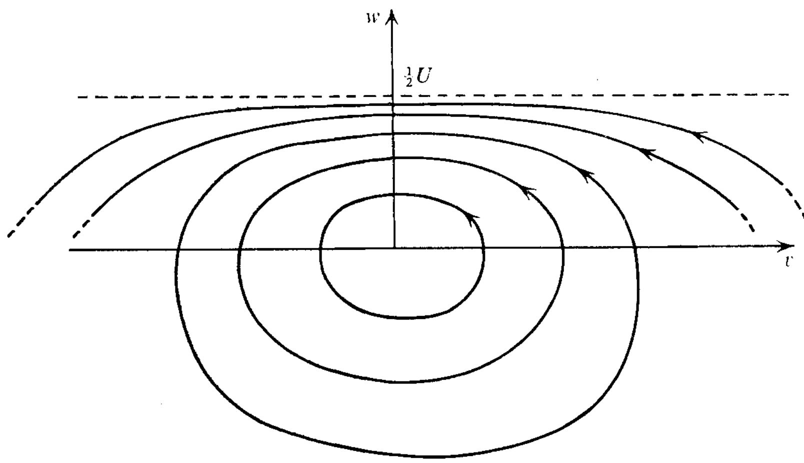

FIGURE 2

Starting on any integral curve at $v = 0$, with $w < \frac{1}{2}U$ of course, we move round it a sufficient number of times to fill the region $0 \leqslant y \leqslant 1$ with similar nodal solutions as in figure 1 (c): each node has to be at a rational point in $0 < y < 1$. For example, the case in which there is one node at $y = \alpha$, $\alpha \neq \frac{1}{2}$, is excluded since on moving from $y = 0$ to $y = \alpha$, v moves along one half of an integral curve in figure 2. For the solution in $\alpha < y \leqslant 1$, v would have to move discontinuously onto a different integral curve.

For a general solution with $m-1$ nodes, as in figure 1 (*c*), we have for each part of the solution, using (22), with $U = U_m$,

$$v(y) = \frac{U_m}{2}\left[y - \frac{q}{m} + \frac{1}{m}\tanh\frac{U_m}{2m\epsilon}\left(\frac{q}{m} - y\right)\right], \quad \frac{q-1}{m} \leqslant y \leqslant \frac{q+1}{m} \quad (q = 1, \ldots, m-1). \tag{24}$$

Using (18) and (24) we get by a straightforward integration the dissipation function for the complete solution with $(m-1)$ nodes:

$$\phi_m = \frac{U_m^2}{8m^2}\left[\tanh\frac{U_m}{2m^2\epsilon} - \tfrac{1}{3}\tanh^3\frac{U_m}{2m^2\epsilon}\right] - \frac{\epsilon U_m}{4}\left[2\tanh\frac{U_m}{2m^2\epsilon} - 1\right] \sim \frac{U_m^2}{12m^2} - \frac{\epsilon U_m}{4}, \tag{25}$$

with exponentially small errors. With (25), (19) gives U_m as

$$U_m \sim 2\times 3^{\frac{1}{2}}m[1+\tfrac{27}{16}\epsilon^2m^2]^{\frac{1}{2}} - \tfrac{9}{2}\epsilon m^2 \tag{26}$$

and hence on substituting this back into (25) we have

$$\phi_m \sim 1 - \epsilon\{2\times 3^{\frac{1}{2}}m[1+\tfrac{27}{16}\epsilon^2m^2]^{\frac{1}{2}} - \tfrac{9}{2}\epsilon m^2\}. \tag{27}$$

The case with no internal nodes has $m = 1$, figure 1 (*a*), and that with one internal node has $m = 2$, figure 1 (*b*), and so on. From (27) we see that for maximum dissipation $m = 1$ and so $\phi_{\max} = \phi_1 \sim 1 - 2\times 3^{\frac{1}{2}}\epsilon$ with the corresponding $U_1 \sim 2\times 3^{\frac{1}{2}} - \frac{9}{2}\epsilon$ from (26) as the minimum U. For minimum dissipation $\phi = 0$ with the corresponding maximum $U = 1/\epsilon$. The extreme cases when $U = O(1/\epsilon)$ are not covered, of course, by (25)–(27) since we would require $m = O(1/\epsilon)$, in which case the distance between each node is $O(\epsilon)$. In these cases a different singular perturbation solution, which we now give, is required.

When $m = O(1/\epsilon)$ we have $U = O(1/\epsilon)$, $|v| \ll 1$ and hence $\phi \ll 1$. To investigate this situation and prove that there is a finite maximum number of possible solutions the appropriate expansions are easily shown to be

$$\left.\begin{aligned} U &= (1/\epsilon) + u_1 + \epsilon u_2 + \ldots, \quad \xi = y/\epsilon + \ldots, \\ v &= \epsilon^{\frac{1}{2}}v_0(\xi) + \epsilon v_1(\xi) + \epsilon^{\frac{3}{2}}v_2(\xi) + \ldots. \end{aligned}\right\} \tag{28}$$

Substitution of (28) into (14) and (15) after a little algebra gives

$$\left.\begin{aligned} v(y) &= \epsilon^{\frac{1}{2}}a\sin\frac{y}{\epsilon}\left(1 - \frac{\epsilon a^2}{12}\right) - \epsilon\frac{a^3}{3}\sin\frac{2y}{\epsilon}\left(1 - \frac{\epsilon a^2}{12}\right) + O(\epsilon^{\frac{3}{2}}), \\ U &= (1/\epsilon) - \tfrac{1}{2}a^2 + O(\epsilon), \end{aligned}\right\} \tag{29}$$

where a is an arbitrary constant. An expansion in ϵ for ξ is necessary so that secular terms can be suppressed: here they would have arisen in the $v_2(\xi)$ in (28). From (29) we see that for v to satisfy the boundary conditions in (14) we must have as a first approximation

$$\sin(y/\epsilon) = 0 \quad \text{at} \quad y = 0, 1, \quad \text{giving} \quad 1/\epsilon = m\pi,$$

and so for small ϵ the maximum m is

$$M = [1/\pi\epsilon], \tag{30}$$

as found by Burgers (1939).

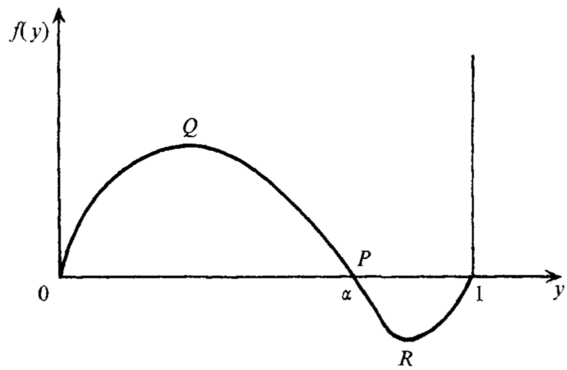

FIGURE 3

Note that if m decreases ϕ_m increases at the expense of the mean flow U_m, which decreases. To proceed further with the type of eddy or turbulent motions which result from an initial disturbance we must consider the unsteady problem. We shall indicate that in the progressive stage ϕ increases at the expense of U. That is, from an unsteady-state point of view we progress from $\phi = 0$ to the ultimate state $\phi = \phi_1$ with U going from $U = 1/\epsilon$ to $U = U_1$. This is in agreement with Lee (1971), who showed, with the quasi-steady form of the equations, that there is an ultimate steady state. The results of Case & Chiu (1969) imply that the opposite is possible and that there is a progression from $\phi = \phi_1$ to $\phi = \phi_2$ to $\phi = \phi_3$ and so on, with a corresponding increase in U.

3. Unsteady problem

We return to (5) and (6) with (7) and consider the initial turbulent velocity distribution $f(y)$ to be such that $0 < \phi(0) = \int_0^1 f^2(y)\,dy < \phi_{\max} = \phi_1$ and U_0 to lie between the minimum and maximum U. To be specific let us take as an example the form illustrated in figure 3, where $0 < \alpha < 1$ is not a rational number. Because of the nonlinear term $2vv_y$ in (5) that part of the wave for which $f'(y) < 0$, that is QPR in figure 3, steepens for $t > 0$ and in the absence of the ϵv_{yy} term shocks develop. The effect of ϵv_{yy} is to smooth out such discontinuities over a distance of $O(\epsilon)$. Its effect away from such shocks is diffusive with a long time scale of $O(1/\epsilon)$.

For times of $O(1)$ the solutions, except at the actual shocks, to $O(1)$ in ϵ are governed by (7) and (5) with the right side zero, namely

$$v_t + 2vv_y - Uv = 0, \quad v(y,0) = f(y), \tag{31}$$

with U given by (6) on solving (31) for $v(y,t;U)$.

In characteristic form (31) becomes

$$y_\sigma = 2v, \quad t_\sigma = 1, \quad v_\sigma - Uv = 0, \tag{32}$$

the last of which shows that v increases with time since $U > 0$. With

$$w(t) = \exp\int_0^t U(\tau)\,d\tau, \quad T(t) = \int_0^t w(\tau)\,d\tau, \tag{33}$$

the solution of (32) may be written parametrically as

$$v(y,t) = f(y_0)\,w(t), \quad y = y_0 + 2f(y_0)\,T(t). \tag{34}$$

Until shocks appear the dissipation $\phi(t)$ is, since $y_0 = 1$ when $y = 1$ ($f(1) = 0$),

$$\begin{aligned}\phi(t) = \int_0^1 v^2\,dy &= \int_0^1 f^2(y_0)\,w^2(t)\,[1+2f'(y_0)\,T(t)]\,dy_0 \\ &= w^2(t)\,\phi(0) + w^2(t)\,T(t)\,[\tfrac{2}{3}f^3(y_0)]_0^1 \\ &= w^2(t)\,\phi(0).\end{aligned} \tag{35}$$

Since $w(t) > 0$ and $dw/dt = U(t)\,w(t) > 0$, because $U > 0$, $w(t)$ increases with time and hence, from (35), $\phi(t)$ increases continuously with time and consequently $U(t)$ from (6) must decrease with time. The mean flow $U(t)$ is the only source of energy for the turbulent motion. The single equation for U is obtained from (6) on using (35) with (33) as

$$\phi^2(0)\exp 2\int_0^t U(\tau)\,d\tau = 1 - U_t - \epsilon U.$$

Taking logarithms and differentiating we get

$$U_{tt} - (2U-\epsilon)\,U_t + 2U(1-\epsilon U) = 0. \tag{36}$$

In the U_t, U phase plane there are two singular points $U_t = 0 = U$ and $U_t = 0$, $U = 1/\epsilon$, the latter being the one of interest in this problem. It is an unstable saddle point.

If we use the quasi-steady-state equation for U the decrease in U with an increase in ϕ is immediately obvious: the equation for $U(t)$ is easily solved and shows exponential decay.

Until shocks appear the point $y = \alpha$, at which $v(y,t) = 0$, does not move in times of $O(1)$. Shocks start, on the $f'(y) < 0$ region of the initial wave, when y in (34) ceases to be a single-valued function of y_0, that is, when $t = t_c$, where t_c is the least value satisfying $2f'(y_0)\,T(t_c) = -1$. For $t > t_c$ shocks grow and propagate. Now $\phi(t)$ ceases to be given by (35) although it still increases: in the shock vicinity ϵv_{yy} is, in fact, of $O(1/\epsilon)$. A specific equation for ϕ is derived below.

For $t > t_c$ let there be a shock at $y = y_s(t)$ and across it let $v(y,t)$ change discontinuously from $v_1(t)$ to $v_2(t)$. Equation (31) implies that the shock propagation speed is

$$\frac{dy_s(t)}{dt} = \frac{\int_{v_1}^{v_2} 2v\,dv}{v_2 - v_1} = v_1 + v_2. \tag{37}$$

Further, each of v_1 and v_2 must lie on the solution (34) when $y = y_s$. Thus, unless $v_1 + v_2$ is zero the shock will propagate. A typical solution using $v(y,0)$ as in figure 3 is as illustrated in figure 4, where we have, for example, $v_1 + v_2 > 0$, which means that the shock moves to the right. The specific analytical details

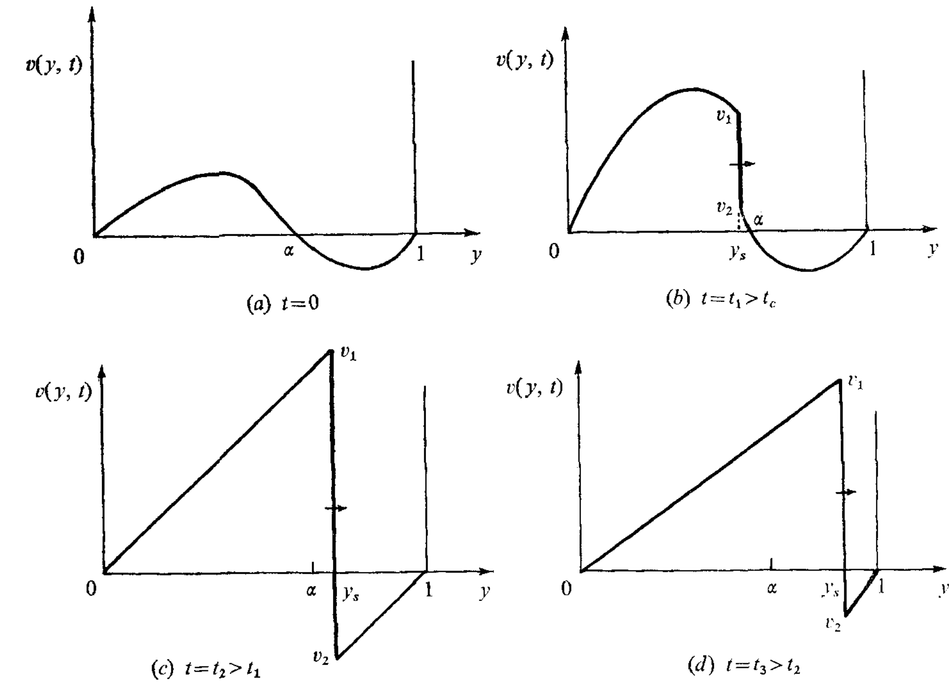

FIGURE 4

for a general initial distribution are not required here but they could be obtained using (34) and (37) with $v_1 = v_2$ at $t = t_c$. The point of relevance here is that the original point $y = \alpha$ where $v = 0$ in figure 3 is overtaken in times of $O(1)$ by a shock as in figures 4 (c) and (d). Ultimately the shock approaches $y = 1$ and $v_2 \to 0$.

To demonstrate that the ultimate state is that with a single steep region at $y = 0$ or $y = 1$ (figure 1 (a)), that is, the one to which the solution in figure 4 is tending, we consider the stability of the shock solution equivalent to figure 1 (b) as $\epsilon \to 0$, namely

$$v(y) = \begin{cases} \tfrac{1}{2}U_2 y & (0 \leqslant y < y_s(0) = \tfrac{1}{2}), \\ \tfrac{1}{2}U_2(y-1) & (\tfrac{1}{2} = y_s(0) < y \leqslant 1), \end{cases} \tag{38}$$

as in figure 5 (a). In (38), $v(y)$ is $v_2(y)$ and U_2 in (24) and (26) respectively as $\epsilon \to 0$. Here the shock is at $y_2 = \tfrac{1}{2}$ and the shock values are initially $v_1 = -v_2 = \tfrac{1}{4}U_2$. At time $t = 0$ we introduce a disturbance into the solution which we take, for simplicity only, as a vertical translation of the shock a distance δ as in figure 5 (b), and consider the initial-value problem

$$\left.\begin{aligned} v(y,0) &= \begin{cases} \tfrac{1}{2}U_2(1+\delta)y & (0 \leqslant y < \tfrac{1}{2}), \\ \tfrac{1}{2}U_2(y-1)(1-\delta) & (\tfrac{1}{2} < y \leqslant 1), \end{cases} \\ U(0) &= U_0 = U_2, \quad \text{say.} \end{aligned}\right\} \tag{39}$$

Here the initial shock values are

$$\left.\begin{aligned} &v_1(0) = \tfrac{1}{4}U_2(1+\delta), \quad v_2(0) = -\tfrac{1}{4}U_2(1-\delta), \\ &[dy_s/dt]_{t=0} = \tfrac{1}{2}U_2\delta, \quad y_s(0) = \tfrac{1}{2}. \end{aligned}\right\} \tag{40}$$

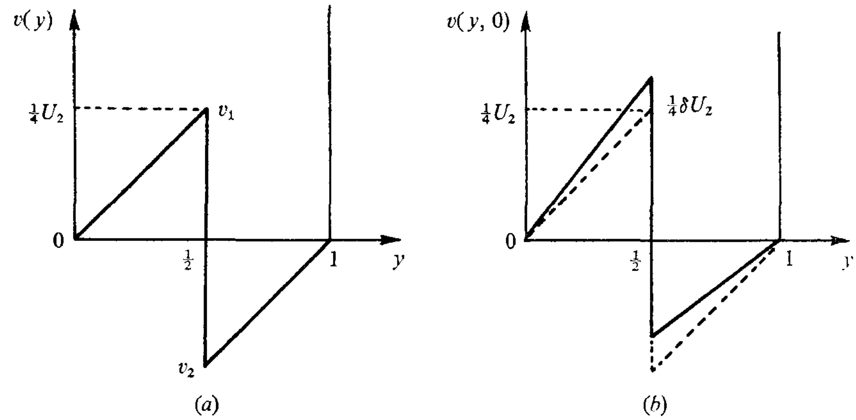

FIGURE 5

For $t > 0$ the shock moves, in this case, towards $y = 1$, and $v_1(t)$ and $v_2(t)$ satisfy (34) with $y = y_s$; that is,

$$\left.\begin{aligned} v_1(t) &= \tfrac{1}{2}U_2(1+\delta)y_0 w(t), \quad v_2(t) = \tfrac{1}{2}U_2(y_0-1)(1-\delta)w(t), \\ y_s(t) &= y_0 + \tfrac{1}{2}U_2(1+\delta)y_0 T(t) = y_0 + \tfrac{1}{2}U_2(1-\delta)(y_0-1)T(t), \end{aligned}\right\} \tag{41}$$

which gives

$$\left.\begin{aligned} v_1(t) &= \tfrac{1}{2}y_s(t)\,w(t)\,U_2(1+\delta)\,[1+\tfrac{1}{2}U_2(1+\delta)\,T(t)]^{-1}, \\ v_2(t) &= \tfrac{1}{2}[y_s(t)-1]\,w(t)\,U_2(1-\delta)\,[1+\tfrac{1}{2}U_2(1-\delta)\,T(t)]^{-1}. \end{aligned}\right\} \tag{42}$$

The shock position $y_s(t)$ is given by the solution of (37) with $v_1(t)$ and $v_2(t)$ as in (42). It is convenient to use $T(t)$ in place of t, in which case the equation for y_s from (37) becomes

$$\frac{1}{w(t)}\frac{dy_s}{dt} = \frac{dy_s}{dT} = v_1(t)+v_2(t) = \frac{U_2[(y_s-\frac{1}{2})\{1+(1-\delta^2)\,U_2T\}+\frac{1}{2}\delta]}{[1+(1+\delta)\,U_2T]\,[1+(1-\delta)\,U_2T]}$$

with $y_s(0) = \frac{1}{2}$. The solution is

$$y_s(T) = \tfrac{1}{2} + (2\delta)^{-1}\,[\{1+2U_2T+(1-\delta^2)\,U_2^2T^2\}^{\frac{1}{2}} - \{1+(1-\delta^2)\,U_2T\}], \tag{43}$$

with $T(t)$ from (33), which involves the undetermined function $U(t)$. As t increases so also does $T(t)$ and so y_s increases from $y_s(0) = \frac{1}{2}$, and if (43) held for all times the shock would reach $y = 1$ in a finite time but with a finite velocity since from (42), if $y_s \to 1$, $v_2(t) \to 0$ and $v_1(t)$ is finite. In the process, of course, $\phi(t)$ is continually increasing with $U(t)$ decreasing. When the shock is in the neighbourhood of $y = 1$, however, (31) is not valid since in the vicinity of the shock (at all positions) ϵv_{yy} is not of $O(\epsilon)$ but in fact of $O(1/\epsilon)$. Inclusion of the ϵv_{yy} term in (31) when the shock is not near $y = 1$ simply smooths it out over distances of $O(\epsilon)$ in the usual way. When the shock is near $y = 1$, however, this term introduces a kind of boundary-layer cushion which brings the shock to rest in a distance of $O(\epsilon)$ and in a time of $O(\epsilon)$ and the ultimate state is $v_1(y)$ since $\phi(t)$ simply grows to its maximum ϕ_1. A demonstration of the cushion effect is given in the appendix.

To complete the mathematical solution (42) and (43) we would require $U(t)$. To obtain the governing equation we need the expression for $\phi(t)$ in such a moving-shock situation. In the vicinity of the shock we introduce into (5) new singular perturbation variables

$$\xi = \epsilon^{-1}[y_s(t)-y], \quad t = t, \quad v = v_1+(v_2-v_1)\,V(\xi), \tag{44}$$

where v_1 and v_2 are given by (41). On substituting (44) into (5) and using (37) for dy_s/dt we get

$$V_{\xi\xi} = (v_2-v_1)\,(1-2V)\,V_\xi+O(\epsilon). \tag{45}$$

From (44) $V(\xi)$ must satisfy

$$V \to 0 \quad \text{as} \quad \xi \to \infty, \quad V \to 1 \quad \text{as} \quad \xi \to -\infty, \tag{46}$$

since we require $v \to v_1$ as $y \to y_s-$ and $v \to v_2$ as $y \to y_s+$. The solution of (45) with (46) is

$$V(\xi) = \tfrac{1}{2}[1-\tanh\{\tfrac{1}{2}(v_1-v_2)\}(\xi-A)]+O(\epsilon), \tag{47}$$

where A is an undetermined constant which represents a shift of $O(\epsilon)$ in the position of the shock as given by (37) with (41). A discussion of such shock positions correct to $O(\epsilon)$ has been given by Burgers [1950*b*, where (44) and (47) above occur together as Burgers' equation (24); see also Burgers (1964), in which a geometrical procedure is given for constructing solutions], and by Murray (1968) for a general class of nonlinear hyperbolic and parabolic equations.

If we now multiply (5) by v and integrate with respect to y from 0 to 1 we get, in place of (18), using (47) in (44) for v,

$$\begin{aligned}\phi_t-U\phi &= -\epsilon\int_0^1 v_y^2\,dy \\ &= (v_2-v_1)^2\int_{\infty}^{-\infty} V_\xi^2\,d\xi+O(\epsilon) \\ &= -\tfrac{1}{6}(v_1-v_2)^3+O(\epsilon).\end{aligned} \tag{48}$$

Equation (48) is consistent with the steady state when $y_s = \frac{1}{2}$, $v_1 = -v_2 = \frac{1}{4}U_2$ and $U = U_2$, since then $\frac{1}{6}(v_1-v_2)^3 = U_2^3/12{\cdot}4$ and so $\phi = \phi_2 = U_2^3/12{\cdot}4$ as in (25).

The full problem to $O(1)$ for the moving shock thus requires the solution of (6) with (48), namely

$$U_t+\epsilon U = 1-\phi, \quad \phi_t-U\phi = -\tfrac{1}{6}(v_1-v_2)^3, \tag{49}$$

where v_1 and v_2 are given by (41) with $w(t)$ given by (33). The problem of finding $U(t)$ is rather intractable analytically: it does decrease as described above. However, the pattern of the solution for the turbulent flow v is clear without it. The shock situation as in figure 5 (*a*) is unstable and tends to the ultimate state with the shock at rest (see appendix) at $y = 1-O(\epsilon)$ and the configuration as in figure 1 (*a*), the maximum turbulent motion with $\phi = \phi_1$ which corresponds to the minimum mean motion $U = U_1$. Any small disturbance will eventually tend to this state and so also will any steady shock state since shocks at points with $0 < y_s < 1$ are necessarily unstable by a similar analysis. Figure 6 illustrates the subsequent motion for two other possible shock situations: arrows indicate the shock motions. It is clear from figure 6 and the above what happens in general.

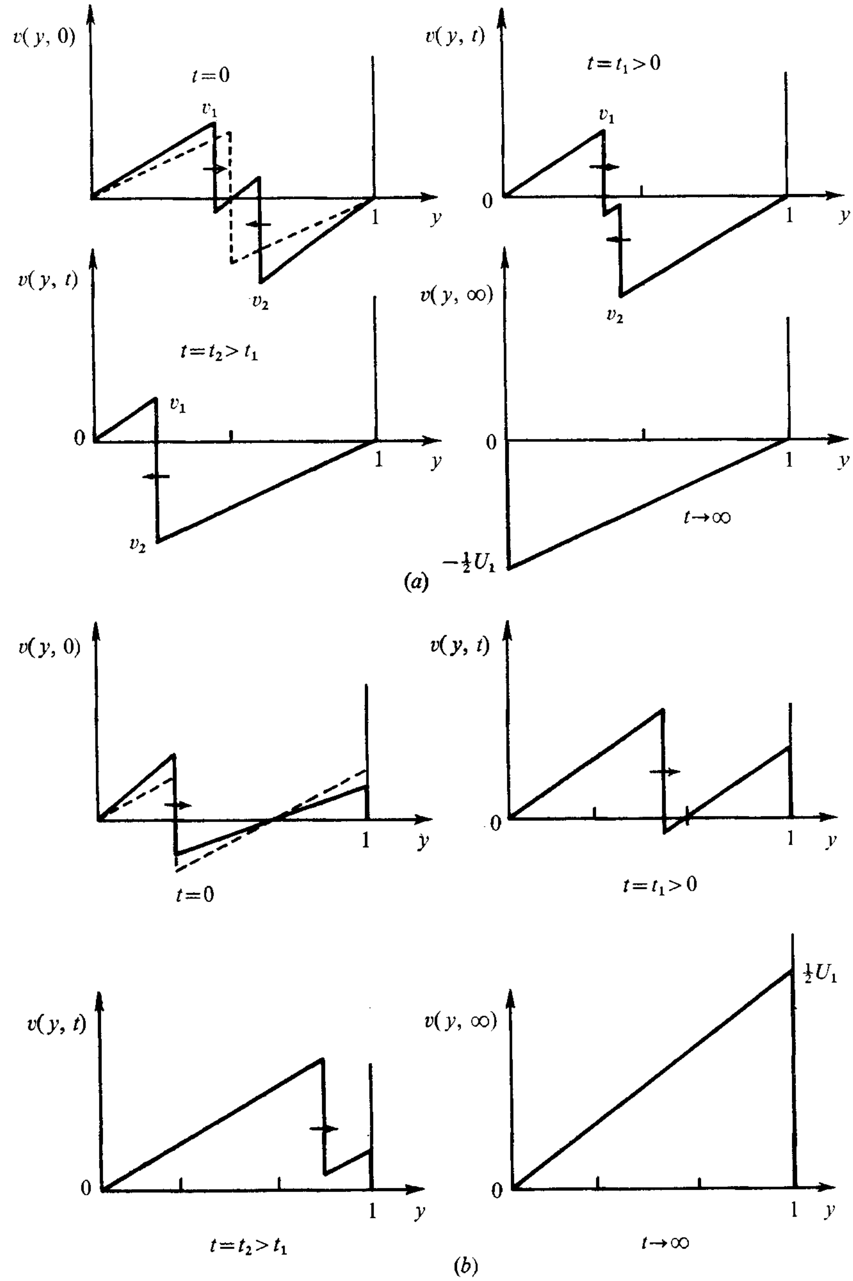

FIGURE 6

While here we considered initial states with a shock somewhere in the interior of the domain, it is of interest to refer to a case mentioned by Burgers (1939), here reproduced as figure 7, where again the point A ($=\alpha$) is different from $y = \frac{1}{2}$. In this case the initial state assumed is constructed from two exact solutions of the time-independent equation (14), both for the same value of U, but with different

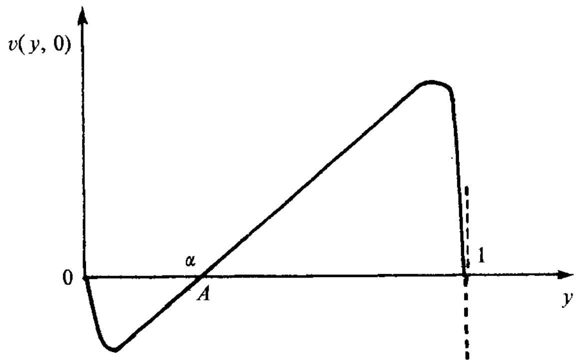

FIGURE 7

values of C. The initial state everywhere satisfies the equation, with the exception of the point A, where the value of $\partial v/\partial y$ is discontinuous (that of v is continuous). The discontinuity in $\partial v/\partial y$ for large U is of order $\exp(-C)$, which in the present notation is something of order $\exp(-U/2\epsilon)$. While it seems probable that the final state of the system will again show a shock layer only at one side of the domain, it may take an extremely long interval of time before this will be reached. By solving Burgers' equation (that is (1) with $U \equiv 0$) with a similar initial condition a time of $O(1/\epsilon)$ is indicated.

4. Conclusions

The above analysis (with the appendix) indicates that the ultimate turbulent state is $v_1(y)$ as in figure 1 (a), with a single large domain of moderate vorticity, accompanied on one side by a narrow vortex sheet, having thickness R^{-1} ($=\epsilon$). When this is described in terms of 'eddies', one may consider the picture of figure 1 (a) as a single 'eddy', but it must be kept in mind that a full spectrum of vortex motion is concentrated in the boundary layer. This spectrum has an amplitude distribution, which was given by Burgers (1939; see also 1948), and it forms an essential feature of the solutions of the model system. The steepening of the velocity profiles represents the excitation of higher Fourier modes while the coalescence of waves, as in figure 6, for example, represents the generation of lower Fourier modes. Thus, like other nonlinear systems, the model possesses a mechanism for the production of vortex motion of smaller and smaller wavelengths, but it has the peculiarity of sweeping all strong vortex motion to one side. Burgers (1950a) has indicated that this result may be generalized to cases of two-dimensional motion, where domains of smooth flow can develop, separated by vortex sheets. Although this feature is not usually observed in turbulent flow, Burgers (1950a) believes that there may be found cases in which its presence can be assumed. Leaving this aside, it appears that Burgers' model, in consequence of its simplificity as compared with the full hydrodynamic equations, has retained a degree of regularity which is not found in most forms of turbulence.

Appendix

As the shock (a weak solution of (31)) approaches the wall $y = 1$, U and $v_1(t)$ are finite and $v_2(t) \approx 0$. To demonstrate the boundary-layer cushion effect of the wall with the minimum of analysis we consider a simpler problem which retains all of the required qualitative features, namely (5), that is,

$$v_t + 2vv_y - Uv = \epsilon v_{yy}, \tag{A 1}$$

with U constant (see below) and

$$\left.\begin{aligned} v(y,0) &= \begin{cases} U & (y < 0), \\ 0 & (y > 0), \end{cases} \\ v(\beta, t) &= 0, \quad v(-\infty, t) = U \end{aligned}\right\} \tag{A 2}$$

as in figure 8, where here we have taken the wall to be $y = \beta$. As the shock approaches the wall we must stretch the co-ordinate perpendicular to the shock in the usual way (see (44)) by writing $\xi = \epsilon^{-1}(y-\beta)$, in which case (A 1) becomes

$$\epsilon v_t = v_{\xi\xi} - 2vv_\xi + O(\epsilon),$$

which, since ϵv_t must be of $O(1)$ here, is to $O(1)$ simply Burgers' equation. We thus see that if U were not constant but had $O(1)$ variations in times of $O(1)$ it could in fact be taken as $U(t)$ as before since it would not affect the cushion phenomenon, which as we shall see occurs in times of $O(\epsilon)$. For simplicity, however, we take it to be constant and the problem reduces to solving (to $O(1)$ in ϵ) the last equation, namely Burgers' equation

$$v_t + 2vv_y = \epsilon v_{yy} \tag{A 3}$$

with conditions (A 2). The usual transformation

$$v(y,t) = -\epsilon\partial[\log \psi(y,t)]/\partial y \tag{A 4}$$

reduces (A 3) with (A 2) to

$$\left.\begin{aligned} &\psi_t = \epsilon\psi_{yy}, \\ &\psi(y,0) = \begin{cases} e^{-Uy/\epsilon} & (y < 0), \\ 1 & (y > 0), \end{cases} \\ &\psi_y(\beta,t) = 0, \quad \psi(y,t) \sim e^{-Uy/\epsilon} \quad \text{as} \quad y \to -\infty, \end{aligned}\right\} \tag{A 5}$$

the continuous solution of which is obtained by Laplace transforms. The transform solution is

$$\left.\begin{aligned} \bar{\psi}(y,p) &= \frac{1}{p} + \frac{U\,e^{-\lambda\beta}}{p(\epsilon\lambda - U)} \cosh\lambda(y-\beta) \quad (y > 0), \\ \lambda &= (p/\epsilon)^{\frac{1}{2}}. \end{aligned}\right\} \tag{A 6}$$

For convenience write

$$\epsilon p = z, \tag{A 7}$$

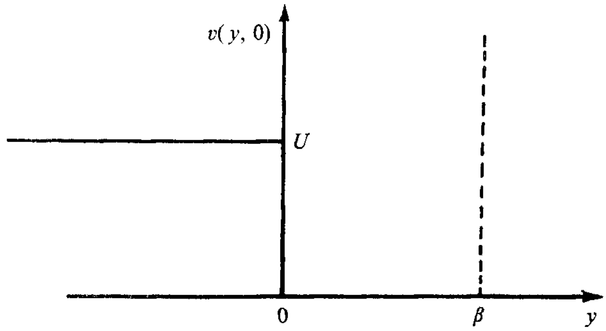

FIGURE 8

in which case the inversion of (A 6) gives the solution to (A 5) in the form

$$\begin{aligned}\psi(y,t) &= 1+\frac{U}{2\pi i}\int_L \frac{\exp[\epsilon^{-1}(zt-z^{\frac{1}{2}}\beta)]}{z(z^{\frac{1}{2}}-U)}\cosh\left(\frac{y-\beta}{\epsilon}\right)z^{\frac{1}{2}}\,dz\\ &= 1+\frac{U}{4\pi i}\int_L \frac{\exp[\epsilon^{-1}f(z)]+\exp[\epsilon^{-1}g(z)]}{z(z^{\frac{1}{2}}-U)}\,dz, \qquad \text{(A 8)}\end{aligned}$$

where L is the Bromwich contour and

$$f(z) = zt-z^{\frac{1}{2}}(2\beta-y), \quad g(z) = zt-z^{\frac{1}{2}}y. \qquad \text{(A 9)}$$

A steepest-descents evaluation of (A 8) for $\epsilon \ll 1$ and $y \approx \beta$, the case of interest here, gives

$$\psi(y,t) \sim 1+2\exp[U\epsilon^{-1}(Ut-\beta)]\cosh[U\epsilon^{-1}(y-\beta)]+\ldots,$$

and finally from (A 4)

$$v(y,t) \sim \frac{U\exp[U\epsilon^{-1}(Ut-\beta)]\sinh[U\epsilon^{-1}(\beta-y)]}{\frac{1}{2}+\exp[U\epsilon^{-1}(Ut-\beta)]\cosh[U\epsilon^{-1}(\beta-y)]}+\ldots \quad (\epsilon\to 0). \qquad \text{(A 10)}$$

For $t \gg 1$, equation (A 10) (or (A 8)) gives the ultimate steady form near $y = \beta$ as

$$v(y,t) \sim U\tanh[U\epsilon^{-1}(\beta-y)], \qquad \text{(A 11)}$$

which is the steady singular perturbation part of $v(y)$ near the boundary when the appropriate values are given in (22).

When y is not near $y = \beta$, equation (A 10) becomes the usual shock transition solution

$$v(y,t) \sim \frac{U\exp[U\epsilon^{-1}(Ut-y)]}{1+\exp[U\epsilon^{-1}(Ut-y)]}+\ldots. \qquad \text{(A 12)}$$

The shock position is given at any time by $Ut-y_s = 0$ – that is, when the exponents in (A 12) are zero: this is the point where, from (A 12), $v(y_s,t) = \frac{1}{2}U$.

The actual shocks in (A 10) and (A 12) are obtained on letting $\epsilon\to 0$. From (A 12) the shock is at $y_s(t) = Ut$ until $y_s = Ut \approx \beta$, in which case (A 10) must be used in place of (A 12). In this case, however, there is essentially no shock. If we use the point where $v(y,t) = \frac{1}{2}U$ as an indication of the 'shock' position we obtain it and its speed of propagation from (A 10). Setting $v(y,t) = \frac{1}{2}U$, $y = y_s$ in (A 10) we get

$$\exp[-U\epsilon^{-1}(Ut-\beta)] \doteqdot \exp[U\epsilon^{-1}(\beta-y_s)]-3\exp[-U\epsilon^{-1}(\beta-y_s)], \qquad \text{(A 13)}$$

from which, for $Ut-\beta > 0$, we get (or more simply from (A 11))

$$y_s \approx \beta - (\epsilon/2U)\log 3. \tag{A 14}$$

From (A 13), on differentiation, the 'shock' speed is

$$\left.\begin{aligned} \frac{dy_s}{dt} &\doteqdot \frac{U\exp[-U\epsilon^{-1}(Ut-\beta)]}{\exp[U\epsilon^{-1}(\beta-y_s)]+3\exp[-U\epsilon^{-1}(\beta-y_s)]},\\ &\to \begin{cases} U & \text{if} \quad Ut<\beta \quad (y_s<\beta),\\ O & \text{if} \quad Ut>\beta, \end{cases} \end{aligned}\right\} \tag{A 15}$$

from which we see that the shock as it approaches the wall is brought to rest exponentially in a time $t = O(\epsilon)$ in a distance of $O(\epsilon)$ from $y = \beta$.

Returning to the problem in the main text we have $\beta = 1$ and $v_1(t)$ in place of U in (A 10), (A 13) and (A 15) as first approximations to the turbulent velocity, the shock position and its speed of propagation near the boundary. The steady-state singular form near $y = 1$ is also obtained. In the limit as $\epsilon \to 0$ equation (35) holds and ϕ eventually tends to its maximum ϕ_1 in (25).

REFERENCES

Burgers, J. M. 1939 Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion. *Proc. Acad. Sci. Amst.* **17**, 1–53.

Burgers, J. M. 1948 A mathematical model illustrating the theory of turbulence. *Adv. Appl. Mech.* **1**.

Burgers, J. M. 1950*a* The formation of vortex sheets in a simplified type of turbulent motion. *Proc. Acad. Sci. Amst.* **53**, 122–133.

Burgers, J. M. 1950*b* Correlation problems in a one-dimensional model of turbulence. *Proc. Acad. Sci. Amst.* **53**, 247–260.

Burgers, J. M. 1964 Statistical problems connected with the solution of a nonlinear partial differential equation. In *Nonlinear Problems of Engineering* (ed. W. F. Ames), pp. 123–137. Academic.

Case, K. M. & Chiu, S. C. 1969 Burgers' turbulence models. *Phys. Fluids*, **12**, 1799–1808.

Lee, J. 1971 Burgers' turbulence model for a channel flow. *J. Fluid Mech.* **47**, 321–335.

Murray, J. D. 1968 Singular perturbations of a class of nonlinear hyperbolic and parabolic equations. *J. Math. & Phys.* **47**, 111–133.